Simpl. set combinatorics for ∞-cats.

- Dennis Chen 5/16/2023

These are some notes on important but technical combinatorial arguments used in Kerodon (by Jacob Lurie).
These arguments are typically used to prove certain maps of sSets are cofibrations (analyze (left/right/mid)).

Typically their statements, while looking difficult, follow some simple 1-categorical intuition about composing many morphisms together.

These notes are an attempt to identify and illuminate these arguments along with badly drawn pictures/diagrams.
The arguments are ~~blatantly stolen from~~ copied from Kerodon.

1). Kerodon Lemma 3.1.2.11:

We can factor the Leibnitz product

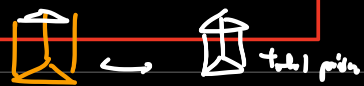
$$\begin{array}{ccc} \partial\Delta^n & & \{1\} = \Lambda^1_1 \\ \downarrow & \hat{\times} & \downarrow \\ \Delta^n & & \Delta^1 \end{array} \quad \text{as}$$

$$(\Delta^1 \times \partial\Delta^n) \coprod_{\{1\}\times\partial\Delta^n} \{1\}\times\Delta^n = X(0) \xrightarrow{\subseteq} X(1) \xrightarrow{\subseteq} X(2) \ldots \xrightarrow{\subseteq} X(n+1) = \Delta^1 \times \Delta^n$$

such that for $0 \le i \le n$, we have a pushout

$$\begin{array}{ccc} \Lambda^{n+1}_{i+1} & \longrightarrow & X(i) \\ \downarrow & & \downarrow \\ \Delta^{n+1} & \longrightarrow & X(i+1) \end{array}$$

(and thus $X(i) \to X(i+1)$ are right anodyne!) htpy eq. of



Pf: (Kerodon)

Let $\theta_i : \Delta^{n+1} \longrightarrow \Delta^1 \times \Delta^n$ denote the map

$$\theta_i := \begin{cases} (0, j) & \text{if } j \le i \\ (1, j-1) & \text{if } j > i \end{cases}$$

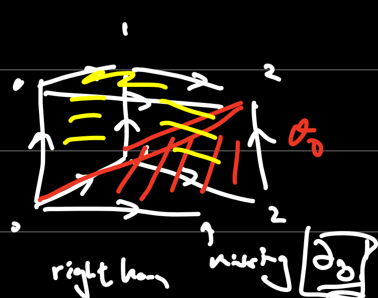
$\Delta^2 \xrightarrow{\theta_0} \Delta^1 \times \Delta^1$

$\Delta^2 \xrightarrow{\theta_1} \Delta^1 \times \Delta^1$

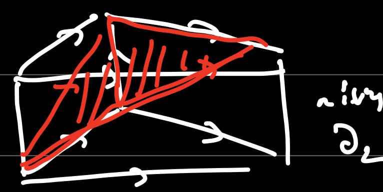
These θ_i are the $(n+1)$ Δ^{n+1}'s that lie in $\Delta^1 \times \Delta^n$!

For $\Delta^3 \xrightarrow{\theta_i} \Delta^1 \times \Delta^2$:

θ_0 :

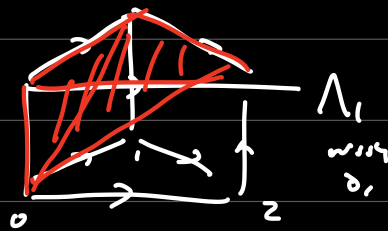


θ_1



θ_2

right horn



Note $\Delta^1 \times \Delta^n$ is the union of all of the θ_i! $\bigcup_{i=0}^{n+1} \mathrm{Im}\,\theta_i = \Delta^1 \times \Delta^n$

Now let

$$X(0) := (\Delta^1 \times \partial\Delta^n) \cup (\{1\} \times \Delta^n)$$

$$X(i+1) := X(i) \cup \mathrm{Im}\,\theta_i .$$

Since $X(n+1) = \bigcup_{i=0}^{n+1} \mathrm{Im}(\theta_i) = \Delta^1 \times \Delta^n$, it is the correct filt'n! $(X(0) \subseteq \dots X(n+1) = \Delta^1 \times \Delta^n)$.

Next: wts

$$\begin{tikzcd} \Lambda^{n+1}_{i+1} \arrow[r, hook] \arrow[d] & X(i) \arrow[d] \\ \Delta^{n+1} \arrow[r] & X(i+1) \end{tikzcd}$$

is pushout.

It's enough to show $\theta_i^{-1}(X(i))$ (i.e. the intersection $X(i) \cap \mathrm{Im}\,\theta_i$) is abstractly equal to Λ^{n+1}_{i+1}.

Regard θ_i as $n+1$ simplex of $\Delta^1 \times \Delta^n$.

WTS the faces $d_j(\theta_i)$ belong to $X(i)$ iff $j \neq i+1$.

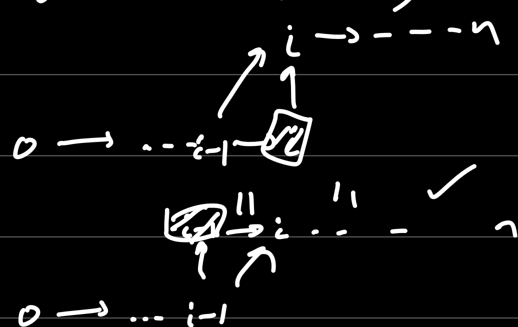
Note if $j \neq i, i+1$ then

$d_j(\theta_i)$ is in $\Delta^1 \times \partial\Delta^n$

n-simplex (j missing).

For $j = i$:

$d_i(\theta_i) = d_i(\theta_{i-1})$



$d_i\,\theta_i$

$\| \checkmark$

$d_i(\theta_{i-1})$

So since $X(i)$ contains $h(\theta_{i-1})$,

$d_i(\theta_i) = d_i(\theta_{i-1})$ in $X(i)$ ✓.

Last: show $d_{i+1}(\theta_i)$ NOT IN $X(i)$:

$= X(0) \cup \theta_1 \cup \dots \sigma_{i-1}$.

$\textcircled{i+1} \to i+1 \to \cdots \to n \leftarrow$ face

$0 \to \cdots \to i$

This is clearly not face in $X(0)$,

$\sigma_1, \sigma_2, \dots \sigma_{i-1}$

$X(0)$ has $\to \to q \; \square \to - -$ mixing there.

or wrong row. ✓

Want to show:

1). Leibnitz closure of
right anodyne w/ mono,
left anodyne w/ mono,
anodyne w/ mono.

2). Lemma 4.4.2.15
which is step in proving

Prop 4.4.2.14: TFAE:

a) $X \to S$ triv Kan fib

b) $X \to S$ left fib + fibers X_s contractible.

c) $X \to S$ right fib + fibers X_s are contractible.

2). Variant:

The Leibniz product

$$\begin{pmatrix} \partial\Delta^n \\ \downarrow \\ \Delta^n \end{pmatrix} \hat{\times} \begin{pmatrix} \Lambda^2_1 \\ \downarrow \\ \Delta^2 \end{pmatrix}$$ has a factorization/ filt.

into

$$X(0) \subseteq \dots X(n) = Y(0) \subseteq \dots \quad Y(n+1) = \Delta^n \times \Delta^2$$

$X(0) = (\Delta^n \times \Lambda^2_1) \cup (\partial\Delta^n \times \Delta^2)$ union

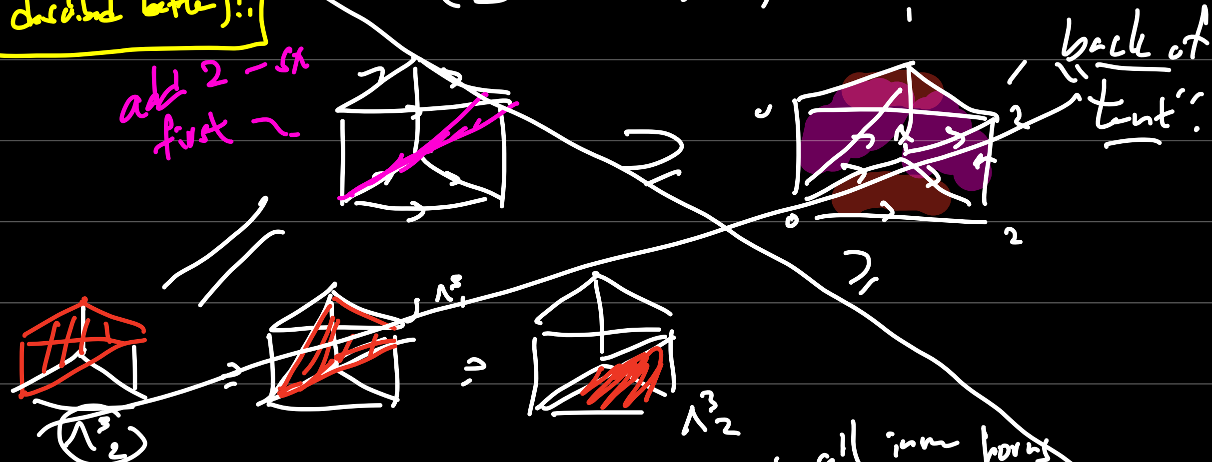
s.t.

each comp. is an inner anodyne mor.

In fact each factors as a large composite of inner anodynes.

Picture. $\begin{pmatrix} \partial\Delta^1 \\ \downarrow \\ \Delta^1 \end{pmatrix} \hat{\times} \begin{pmatrix} \Lambda^2_1 \\ \downarrow \\ \Delta^2 \end{pmatrix}$:

(idea better described better)!!



Idea: can "compose naturally" w/ boundary conditions, all inner horns

$$\underbrace{\Delta^n \times \Lambda^2_1}_{\text{natural commute diagram}} \cup \underbrace{\partial\Delta^n \times \Delta^2}_{\text{prescribed boundary composites}} \longrightarrow \Delta^n \times \Delta^2$$

Pf: We look at certain simplices in $\Delta^m \times \Delta^2$.

Let θ_{ij} be determined by $0 \le i \le j < m$

$$\Delta^{m+1} \xrightarrow{\theta_{ij}} \Delta^m \times \Delta^2 \quad \text{via}$$

$$\theta_{ij}(k) = \begin{cases} (k,0) & 0 \le k \le i \\ (k-1,1) & i+1 \le k \le j+1 \\ (k-1,2) & j+2 \le k \le m+1 \end{cases}$$

Picture of θ_{ij} for $\Delta^1 \times \Delta^2$

Only θ_{00}

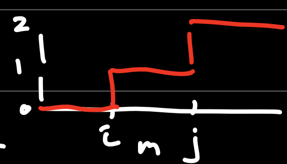
θ_{00}

diagonal 2-simplex.

Let τ_{ij} be Δ^{m+2} simplex in $\Delta^m \times \Delta^2$ def'd by

$$\Delta^{m+2} \xrightarrow{\tau_{ij}} \Delta^m \times \Delta^2 \quad \text{for } 0 \le i \le j \le m:$$

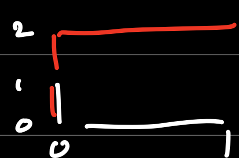
$$\tau_{ij}(k) = \begin{cases} (k,0) & 0 \le k \le i \\ (k-1,1) & i+1 \le k \le j+1 \\ (k-2,2) & j+2 \le k \le m+2 \end{cases}$$



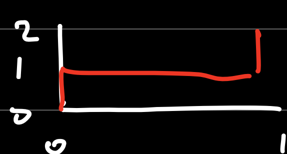
Picture τ_{ij} for $\Delta^1 \times \Delta^2$:

τ_{00}

bottom

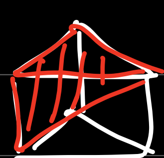


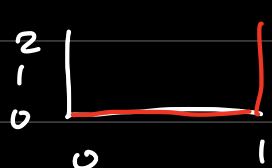
τ_{01}



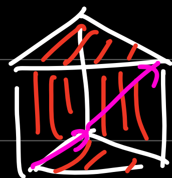
center one

τ_{11}



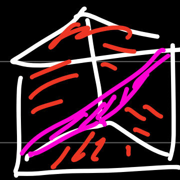


Idea: First add θ_{ij}:



$\xrightarrow{\text{inner horn}}$

$\Lambda^2_1 \hookrightarrow \Delta^2$



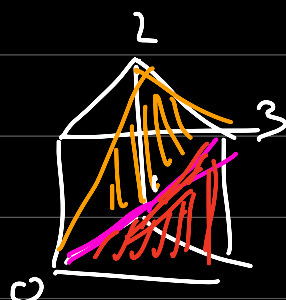
Then add

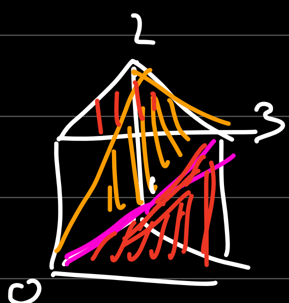
τ_{00}:

$\Lambda^3_1 \hookrightarrow \Delta^3$

τ_{01}:

$\Lambda^2_1 \hookrightarrow \Delta^3$





$\xrightarrow{\text{yellow}}$

last Sx $\Lambda^3_2 \hookrightarrow \Delta^3$

Set $X(0) := \Delta^m \times \Lambda^2_1 \cup \partial\Delta^m \times \Delta^2$ in $\Delta^m \times \Delta^2$

(iso to $\Delta^m \times \Lambda^2_1 \coprod_{\partial\Delta^m \times \Lambda^2_1} \partial\Delta^m \times \Delta^2$).

Inductively let $X(j+1) := X(j) \cup \theta_{0j} \cup \dots \cup \theta_{jj}$

(add in all $m+1$-simplices first in order to put in $m+2$ sx's).

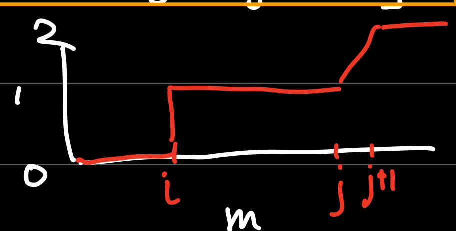
We have chain of inclusions

$$X(j) \subseteq X(j) \cup \theta_{0j} \subseteq \dots \subseteq X(j) \cup \theta_{0j} \dots \cup \theta_{jj} = X(j+1).$$

Each incl. is a pushout:

$$\boxed{\begin{tikzcd} \Lambda^{m+1}_{i+1} \arrow[r] \arrow[d] & X(j) \cup \theta_{0j} \cup \dots \theta_{(i-1)j} \arrow[d] \\ \theta_{ij} \arrow[r] & X(j) \cup \theta_{0j} \dots \theta_{ij} \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}}$$

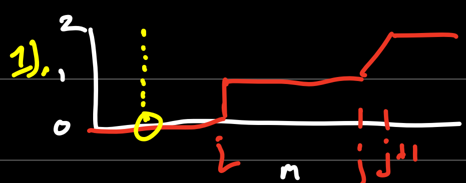
Diagram:



Look at horns Λ^{m+1}_{j+1}: delete one pt except $j+1$'th pt.

Case:

1)

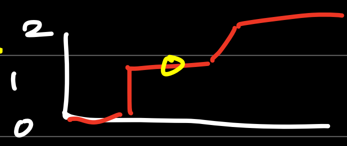


This is in $\partial\Delta^m \times \Delta^2$.

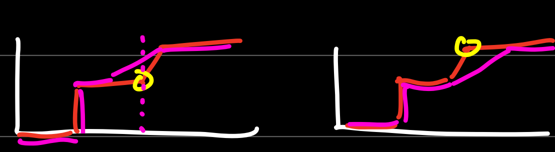
2)

In θ_{i-1j}!

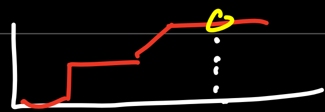
missing one! Λ^{m+1}_{i+1}

3. 

in $\partial\Delta^n\times\Delta^2$

4. 

in $\partial\Delta^n\times\Delta^2$

5.  in $\partial\Delta^n\times\Delta^2$.

Edge cases:

6.  

θ_{00} in $\Delta^n\times\Lambda^2_1$ $\theta_{n-1,n-1}$

Next we do another filt.

$Y(0) := X(n)$ (has all θ_{ij}!, $\Delta^n\times\Lambda^2_1$, and $\partial\Delta^n\times\Delta^2$).

Inductively let

$$Y(j+1) := Y(j)\cup\tau_{0j}\cup\dots\tau_{jj} \quad \text{for } 0\le j\le n.$$

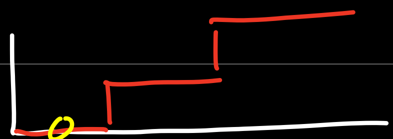
$$Y(j)\subseteq Y(j)\cup\tau_{0j}\subseteq\dots\subseteq Y(j)\cup\tau_{0j}\cup\dots\tau_{jj} = Y(j+1)$$

Each step fits into a pushout

$$\begin{tikzcd}
\Lambda^{n+2}_{i+1} \arrow[r] \arrow[d] & Y(j)\cup\tau_{0j}\cup\dots\cup\tau_{(i-1)j} \arrow[d] \\
\tau_{ij} \arrow[r] & Y(j)\cup\dots\cup\tau_{ij} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

!

Case.

1). 

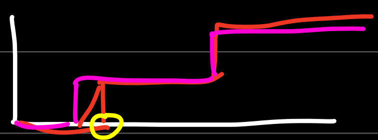
in $\partial\Delta^n \times \Delta^2$

Edge

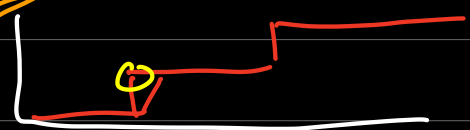
$\Delta^n \times \Lambda^2_1$

not in!! Λ^n_1

missing face

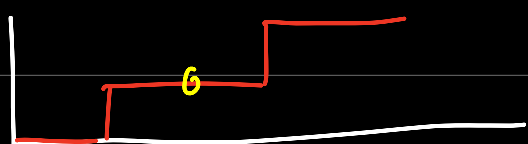
2). 

face of $\tau_{i-1,j}$

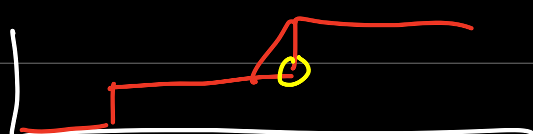


← Missing face

Λ^{n+2}_{i+1}

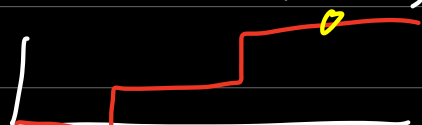
3). 

in $\partial\Delta^m \times \Delta^2$

4). 

This is $\theta_{i,j-1}$

This is θ_{ij}

5). 

in $\partial\Delta^n \times \Delta^2$. $\square$.

3). Natural Isos:

Let $m \geq 0$, $n \geq 2$ be integers. There is a factorization/filt of the Leibniz product $\begin{pmatrix}\partial\Delta^m\\ \downarrow\\ \Delta^m\end{pmatrix} \hat{\times} \begin{pmatrix}\Lambda^n_0\\ \downarrow\\ \Delta^n\end{pmatrix}$ as

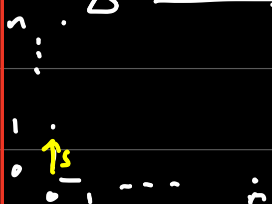
$$(\partial\Delta^m\times\Delta^n)\cup(\Delta^m\times\Lambda^n_0) = X(0) \subseteq \dots \qquad \dots X(t) \overset{(*)}{=} \Delta^m\times\Delta^n$$

s.t. $\forall\ 0 < s \leq t$, $\exists\ q \geq 2$ and $0 \leq p < q$ + pushout

$$(*)\quad \begin{tikzcd} \Lambda^q_p \arrow[r] \arrow[d] & X(s-1) \arrow[d] \\ \Delta^q \arrow[r, "\theta"] & X(s) \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

Moreover if $p=0$ then $\theta: \Delta^2 \to X(s) \subseteq \Delta^m\times\Delta^n$ has

$$\theta(0) = (0,0)$$
$$\theta(1) = (0,1)$$



Rmk: If uses a left horn, θ has first row the yellow one which will be invertible!

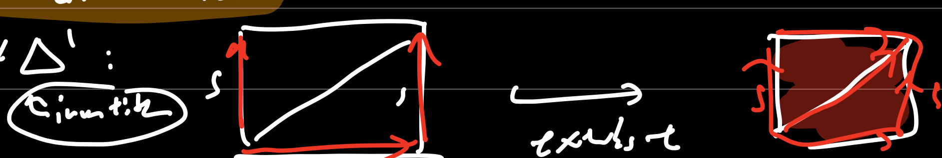
Says we only need $0 \to 1$ in Δ^n to be invertible! Then we can extend

$$\Delta^m\times\Lambda^n_0 \cup \partial\Delta^m\times\Delta^n \longrightarrow \Delta^m\times\Delta^n$$

Even though it has outer horn!!

Low dim idea:

$\Delta^1\times\Delta^1$:



$$\Delta^1\times\Lambda^1_0 \cup \partial\Delta^1\times\Delta^1$$

$\Delta^1 \times \Delta^2$

$\Delta^1 \times \Lambda^2_0 \cup \partial\Delta^1 \times \Delta^2$

Idea: reverse (use 01 in Δ^n to) Λ^n_0 into an inn horn!

ie

Pf: Let σ be nondegen q-sx. in $\Delta^m \times \Delta^n$ (so $\dim(\sigma)$ well defined)

$\sigma : \Delta^q \longrightarrow \Delta^m \times \Delta^n$ rep. by

$(i_0, j_0) < (i_1, j_1) < \dots < (i_q, j_q)$.

Call σ "free" if the comp.

$\Delta^q \xrightarrow{\sigma} \Delta^m \times \Delta^n \twoheadrightarrow \Delta^m$

$\Delta^q \xrightarrow{\sigma} \Delta^m \times \Delta^n \twoheadrightarrow \Delta^n$

are surj,

and there is $0 \leq p < q$ s.t.

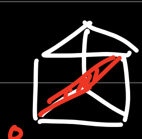
$(i_p, j_p) = (p, 0)$ ~ call this $p(\sigma)$

$(i_{p+1}, j_{p+1}) = (p, 1)$.

Note that this p is then uniquely det'd.

Picture in $\Delta^1 \times \Delta^2$ $p = 0$

Ex:



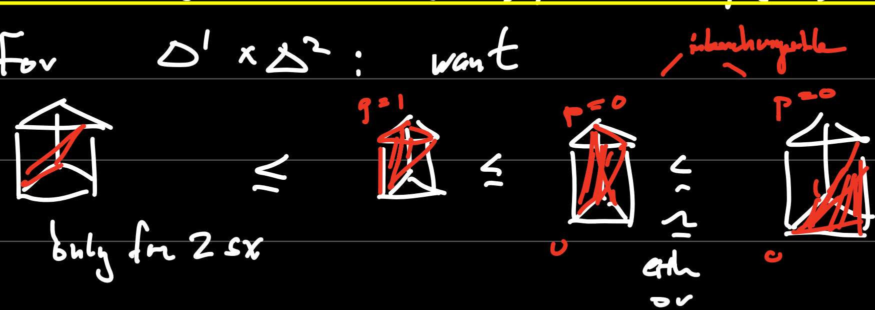
Note that these free guys are the σ_{ij}'s τ_{ij}'s above in $\Lambda^2_1 \to \Delta^2$ arg.

Let $\{\sigma_1, \sigma_2, \dots, \sigma_t\}$ be an enumeration of the free sx's s.t.

A). For $1 \le s \le s' \le t$, we have
$$\dim(\sigma_s) \le \dim(\sigma_{s'})$$

B). If $1 \le s \le s' \le t$ s.t.
$\dim(\sigma_s) = \dim(\sigma_{s'})$, then $p(\sigma_s) \ge p(\sigma_{s'})$

For $\Delta^1 \times \Delta^2$: want



Let $X(0) := (\Delta^m \times \Lambda^n_0) \cup (\partial\Delta^m \times \Delta^n)$

Let $X(s)$ denote the smallest simpl set of $\Delta^m \times \Delta^n$ containing $X(0)$ and $\{\sigma_1, \dots \sigma_s\}$.

So $X(s) = X(0) \cup \sigma_1 \cup \dots \sigma_s$.

Wts $X(0) \subseteq X(1) \dots \subseteq X(t)$ satisfies the conditions. ⊗ & ⊛

First check $X(t)$ is all $\Delta^m \times \Delta^n$. ⊗

Let σ be a nondegenerate sx

$(i_0,j_0) < (i_1,j_1) < \dots < (i_q,j_q)$

Can assume $i_0, \dots i_q$ range all $[0,n]$ and

$(j_0 \dots j_q)$ range on $[1,n]$

as $X(t)$ contains $X(0) = \Delta^n \times \Lambda^n_0 \cup \partial\Delta^n \times \Delta^n$

Thus σ contains some $(p,1)$, choose p small as possible.

<u>Case</u>:

A). σ contains $(p,0)$, so is free! (surj onto Δ^n!) $\checkmark$ clear.

B). σ doesn't contain $(p,0)$. So is

$(0,0) < (1,0) < (2,0) \dots (p-1,0) < (p,1) < (i_{p+1}, j_{p+1}) < \dots (i_q, j_q)$

$< (p,0) <$

Then σ is face of σ' (add in $(p,0)$), which <u>IS</u> free! $\checkmark$

Now we complete pf by verifying $\star$:

$\star$ ie $\forall 0 < s \leq t$, $\exists\, q \geq 2$, $0 \leq p < q$ and pushout

$$\begin{tikzcd} \Lambda^q_p \arrow[r] \arrow[d] & X(s-1) \arrow[d] \\ \Delta^q \arrow[r, "\sigma"] & X(s) \end{tikzcd}$$

and if $p=0$ then $\sigma : \Delta^q \to \Delta^m \times \Delta^n$ has

$\sigma(0) \to \sigma(1) = (0,0) \to (0,1)$.

Fix $0 < s \leq t$.

Let $\sigma = \sigma_s$ be the free sx.

Let $q = \dim \sigma$

$p = p(\sigma)$.

So σ looks like

$$(0,0) < (1,0) \cdots < (p,0) < (p,1) < (i_{p+2}, j_{p+2}) < \cdots < (i_q, j_q).$$

Clearly $X(s) = X(s-1) \cup \sigma$.

We check that $K = \sigma \cap X(s-1)$ is

$\Lambda^q_p \subseteq \Delta^q = \sigma$, so that pullback

$$\begin{tikzcd}
\Lambda^q_p = K \arrow[r] \arrow[d] & X(s-1) \arrow[d] \\
\Delta^q = \sigma \arrow[r, "\sigma"] & X(s)
\end{tikzcd}$$

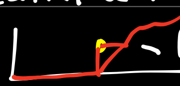
is also a pushout.

Goal: Prove $K = \Lambda^q_p$.

First we show $\Lambda^q_p \subseteq K$.

To do this: we show each face d_i except d_p is in K.

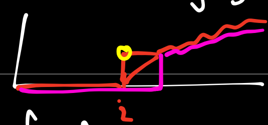
Cases 1).  $i < p$ contained in $\partial\Delta^m \times \Delta^n$.

2). No $i \geq p$. For $i = p+1$:  (i_{p+2}, j_{p+2})

$(p,0)$

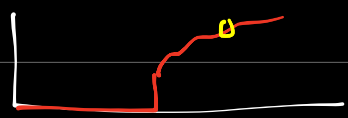
If $j_{p+2} \geq 2$ then missing $j=1$ row: in $\Delta^n \times \Lambda^n_0$.

Else $j_{p+2} = 1$



This is face of free sx σ'! - added previously in $X(s-1)$

b.c. $p(\sigma') > p(\sigma)$! ✓

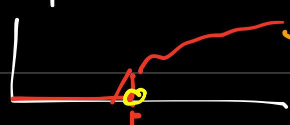
3).  $\leftarrow d_i\sigma$ is lower dim than σ.

Either contained in $X(0)$ or is free, lower dim! So in $X(s-1)$

So $\Lambda^q_p \subseteq X(s-1)$ (and thus is sub of $K = X(s-1) \cap \sigma$.

Done.

Next: show $K \subseteq \Lambda^q_p$. Suffices to show $d_p\sigma \notin K$.



$\tau = d_p\sigma$, $(0,0) < \dots (p-1,0) < (p,1), \dots$

Assume that τ contained in K, for contra.

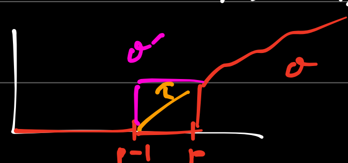
τ not in $X(0)$ b.c. σ is free.

So τ contained in free sx $\sigma' < \sigma$ (as τ itself isn't free).

$\dim(\tau) = q-1 < \dim\sigma' \leq \dim\sigma$ via ordering.

So σ' free + not σ + contains τ!

Must be $(0,0) < (1,0) < \dots (p-1,0) < (p-1,1)$



Then $p(\sigma) > p(\sigma')$, implying $\sigma < \sigma'$

contradicting $\sigma' < \sigma$!! ! ✓

Last: $p=0$ condition follows from

det. of p! $p = \underline{p(\sigma)}$. ✓

ie $(0,0) - (0,1), <\ldots \sigma$

Done!